

NECESSITY CONDITIONS FOR L^p MULTIPLIERS WITH POWER WEIGHTS

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ABSTRACT. It is shown that if multiplier operators are bounded on L^p with weight $|x|^\alpha$ for all functions in the space $\mathcal{S}_{0,0}$ of Schwartz functions whose Fourier transforms have compact support not including 0 and all multiplier functions in a standard Hörmander type multiplier class, then α must satisfy certain inequalities. This is a sequel to a previous paper in which conditions on α that were almost the same were shown to be sufficient for the norm inequality to hold.

1. Introduction. This paper is concerned with conditions that a real number α must satisfy if

$$(1.1) \quad \int_{-\infty}^{\infty} |(mf)^{\vee}(x)|^p |x|^\alpha dx \leq C \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx$$

for all m in a standard Hörmander type multiplier class and all f in $\mathcal{S}_{0,0}$, the Schwartz functions whose Fourier transforms have compact support not including 0. This is a sequel to [5] in which sufficient conditions were obtained on α for (1.1) to hold.

As in [5] we use the multiplier classes $M(s, \lambda)$; for λ a positive integer and $1 \leq s \leq \infty$ the class $M(s, \lambda)$ consists of all m such that

$$B(m, s, \lambda) = \|m\|_\infty + \sup_{r>0} r^{\lambda-1/s} \left[\int_{r<|t|<2r} |m^{(\lambda)}(t)|^s dt \right]^{1/s} < \infty.$$

For the definition of $M(s, \lambda)$ and $B(m, s, \lambda)$ with λ not an integer, see §2. As shown in §7 of [5], these spaces are two sided versions of the multiplier classes $S(s, \lambda)$ in [2] and $WBV_{s,\lambda}$ in [3].

The following was proved in [5]; it is Theorem (1.2) of [5].

THEOREM (1.2). *If $1 < p < \infty$, $1 \leq s \leq \infty$, $\lambda > \max(1/s, |1/p - 1/2|)$ or $\lambda = s = 1$, m is in $M(s, \lambda)$,*

$$\begin{aligned} &\max(-1, -p\lambda, -1 + p(-\lambda + \tfrac{1}{2})) \\ &< \alpha < \min(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s})) \end{aligned}$$

Received by the editors December 20, 1984.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42A45; Secondary 44A35, 26A33, 42A38.

Research supported in part by NSF grant.

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0002-9947/87 \$1.00 + \$.25 per page

and $(\alpha + 1)/p$ is not an integer, then for f in $\mathcal{S}_{0,0}$

$$(1.3) \quad \int_{-\infty}^{\infty} |(mf)^{\vee}(x)|^p |x|^{\alpha} dx \leq CB(m, s, \lambda)^p \int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha} dx,$$

where C is independent of m and f .

The conditions on α in Theorem (1.2) seem peculiar, both because they are complicated and because taking s larger than the minimum of 2 and $p' = p/(p - 1)$ does not increase the range of α . The main result of this paper is the fact that, except possibly for the strictness of the inequalities, these conditions are also necessary. The result is as follows.

THEOREM (1.4). *If $1 < p < \infty$, $1 \leq s \leq \infty$, $\lambda \geq 1/s$ and (1.1) holds for all m in $M(s, \lambda)$ and f in $\mathcal{S}_{0,0}$, then $\alpha > -1$,*

$$\max(-p\lambda, -1 + p(-\lambda + \tfrac{1}{2})) \leq \alpha \leq \min(p\lambda, -1 + p(\lambda + \tfrac{1}{2}), -1 + p(\lambda + 1 - \tfrac{1}{s}))$$

and $(\alpha + 1)/p$ is not an integer.

In at least some cases, the end values of the inequalities for α are included in the values for which (1.1) holds; a theorem of this type is given in §6 of [5]. It should also be remarked that the conclusion that $(\alpha + 1)/p$ is not an integer can be proved from much weaker hypotheses. In fact, as shown in Theorem (1.2), p. 624 of [6], if (1.1) holds for all f in $\mathcal{S}_{0,0}$ for some m that is not a constant almost everywhere, then it follows that $(\alpha + 1)/p$ is not an integer. The proof given here uses the stronger hypothesis and, as a result, is much shorter than the proof in [6].

This paper consists of the proof of Theorem (1.4). §2 contains certain facts needed in later sections primarily concerning fractional derivatives and the multiplier classes $M(s, \lambda)$. These were proved in [5] and are quoted here for convenience. §3 gives some general procedures for generating functions in the classes $M(s, \lambda)$. These are used in §4 to produce specific examples of functions in $M(s, \lambda)$ that prove the upper bounds for α in Theorem (1.4). In §5 the facts that $\alpha > -1$ and $(\alpha + 1)/p$ is not an integer are proved directly while the lower bounds on α are obtained from the upper bounds by a duality argument.

The following definitions and notations will be used throughout this paper. Given real α and $p \geq 1$, we define

$$\|f\|_{p,\alpha} = \left[\int_{-\infty}^{\infty} |f(x)|^p |x|^{\alpha} dx \right]^{1/p}$$

and L_{α}^p as the set of f with $\|f\|_{p,\alpha} < \infty$; as usual, α may be omitted if it is 0. The space $\mathcal{S}_{0,0}$ will be as defined above. For integrable f we define the Fourier transform $\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$ and the inverse Fourier transform

$$\check{f}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{ixt} dt.$$

For general locally integrable f , we define $\hat{f}$ to be the function that satisfies $\int_{-\infty}^{\infty} \hat{f}(x)\phi(x) dx = \int_{-\infty}^{\infty} f(x)\hat{\phi}(x) dx$ for every ϕ in C^∞ with compact support, provided such a function exists. The inverse Fourier transform $\check{f}$ for locally integrable functions is defined analogously. Similarly, the weak derivative of a function f on $(-\infty, \infty)$ is the function f' such that $\int_{-\infty}^{\infty} f(x)\phi'(x) dx = -\int_{-\infty}^{\infty} f'(x)\phi(x) dx$ for every ϕ in C^∞ with compact support, provided such a function exists.

Throughout this paper C will denote constants not necessarily the same at each occurrence. If $g(x)$ is an expression in x , $[g(x)]^\wedge$ will denote the Fourier transform at the point x . For a number p with $1 \leq p \leq \infty$, p' will denote $p/(p-1)$. In addition to the expression $\text{int}(x)$ for the greatest integer less than or equal to x , the traditional $[x]$ will also be used when not ambiguous.

2. Definitions and basic results. For convenience, we list here the definitions and theorems from [5] that will be needed to prove Theorem (1.4). These are various properties of the multiplier classes $M(s, \lambda)$ and fractional derivatives plus one density theorem. For proofs and further discussion, see [5].

We define the operator D^λ by $D^\lambda g(x) = [\check{g}(x)x^\lambda]^\wedge$, where x^λ is taken to be $|x|^\lambda e^{-i\pi\lambda}$ for $x < 0$ and the Fourier transforms are as defined in §1. To define the multiplier classes, choose a function $\psi(x)$ in C^∞ with support in $1/2 < |x| < 2$ such that $\sum_{j=-\infty}^{\infty} \psi(2^{-j}x) = 1$ for $x \neq 0$. Given a function $m(x)$, define $m_j(x) = m(x)\psi(2^{-j}x)$. Then for $1 \leq s \leq \infty$ and $\lambda \geq 0$, the multiplier class $M(s, \lambda)$ is the set of functions m such that $D^\lambda m_j$ is a locally integrable function for every j and

$$(2.1) \quad B(m, s, \lambda) = \|m\|_\infty + \sup_j 2^{j(\lambda-1/s)} \|D^\lambda m_j(x)\|_s < \infty.$$

We will need the following two results concerning the classes $M(s, \lambda)$. The first shows that the classes are independent of the choice of the function ψ ; the second is an imbedding theorem. They are Theorem (2.25) and Theorem (2.12) of [5].

THEOREM (2.2). *If m is in $M(s, \lambda)$, $1 \leq s \leq \infty$, $\lambda > 0$, and ϕ has $[\lambda + 1]$ bounded derivatives and support in $1/2 \leq |x| \leq 2$, then $D^\lambda[m(x)\phi(2^{-j}x)]$ is a locally integrable function and*

$$\|D^\lambda[m(x)\phi(2^{-j}x)]\|_s \leq CA(\phi)2^{j(-\lambda+1/s)}B(m, s, \lambda),$$

where C is independent of m and ϕ and $A(\phi) = \sup_{0 \leq k \leq [\lambda+1]} \|\phi^{(k)}\|_\infty$.

THEOREM (2.3). *If $1 \leq s \leq \infty$, $1 \leq t \leq \infty$, $0 \leq \alpha \leq \lambda$, m is in $M(s, \lambda)$ and one of the following holds:*

- (i) $\alpha - 1/t \leq \lambda - 1/s$, $s > 1$ and $t < \infty$,
- (ii) $\alpha - 1/t \leq \lambda - 1/s$, $s = 1$ and $t = \infty$,
- (iii) $\alpha - 1/t < \lambda - 1/s$,

then m is in $M(t, \alpha)$ and $B(m, t, \alpha) \leq CB(m, s, \lambda)$ with C independent of m .

The following facts about the fractional derivative operator will also be needed. These are respectively Lemma (2.6), Lemma (2.15) and Lemma (2.18) of [5].

LEMMA (2.4). If $f(x)$ is integrable, $f(x) = 0$ for x not in a compact interval I , $\lambda > -1$ and $D^\lambda f(x)$ is a locally integrable function, then there is a C , depending only on λ , such that $D^\lambda f(x) = 0$ for almost every x to the right of I and

$$D^\lambda f(x) = C \int_I \frac{f(t)}{(t-x)^{\lambda+1}} dt$$

for almost every x to the left of I .

LEMMA (2.5). If $0 < \lambda < 1$, f is in L^1 and either the function

$$G^\lambda f(x) = \frac{d}{dx} \int_x^\infty \frac{f(t)}{(t-x)^\lambda} dt,$$

where the derivative is taken in the weak sense, or $D^\lambda f(x)$ is a locally integrable function, then the other is a locally integrable function and there is a nonzero constant C such that $D^\lambda f(x) = CG^\lambda f(x)$.

LEMMA (2.6). If $0 \leq \lambda < 1$, $1 \leq s \leq \infty$, f is integrable, $D^\lambda f$ is a locally integrable function, ϕ is differentiable with $\|\phi'\|_\infty < \infty$ and ϕ has support in a finite interval I , then $D^\lambda(\phi f)$ is a locally integrable function and

$$\|D^\lambda(\phi f)\|_s \leq C(\|\phi\|_\infty + |I| \|\phi'\|_\infty) (\|\chi_I D^\lambda f\|_s + |I|^{-\lambda} \|f\|_s),$$

where C is independent of f and ϕ .

Finally, we shall need the following density result; it is Theorem (1.5) of [5]. The space Q_k in Theorem (2.7) for $k \geq 0$ is the set of functions f in $L^2 \cap L^1_k$ such that $\int_{-\infty}^\infty f(x)x^j dx = 0$ for $0 \leq j \leq k$; the space Q_{-1} is L^2 .

THEOREM (2.7). If $1 < p < \infty$, $\alpha > -1$, k is an integer, $k \geq -2 + (\alpha + 1)/p$, m is bounded, and

$$(2.8) \quad \|(m\hat{f})^\vee\|_{p,\alpha} < C\|f\|_{p,\alpha}$$

for all f in $\mathcal{S}_{0,0}$, then (2.8) is true for all f in $Q_k \cap L^p_\alpha$ with the same C .

3. Lemmas for Theorem (1.4). To prove Theorem (1.4), we need a way to construct examples of functions in $M(s, \lambda)$, and we will need a modified form of (1.3) for these functions. The results needed are Lemmas (3.1), (3.5), (3.14) and (3.22).

LEMMA (3.1). If $0 < \lambda \leq 1$, $|m(x)| \leq A/(1 + |x|^\lambda)$ and $|m'(x)| \leq A|x|^{-\lambda}$, then m is in $M(\infty, \lambda)$ and $B(m, \infty, \lambda) \leq CA$, where C depends only on λ .

For $\lambda = 1$ this is immediate; therefore, assume that $0 < \lambda < 1$. It is sufficient to show that

$$(3.2) \quad \frac{d}{dx} \int_0^\infty \frac{m(x+t)\psi(2^{-j}(x+t))}{t^\lambda} dt$$

has absolute value bounded by $CA2^{-j\lambda}$ for almost every x since Lemma (2.5) will then imply that $D^\lambda m_j(x)$ equals a constant independent of j times (3.2), and $|D^\lambda m_j(x)| \leq CA2^{-j\lambda}$ is what is needed.

To estimate (3.2) for $2^{j-2} < |x| < 2^{j+2}$, $j > 3$, write (3.2) as the sum of

$$(3.3) \quad \frac{d}{dx} \int_0^1 \frac{m(x+t)\psi(2^{-j}(x+t))}{t^\lambda} dt$$

and

$$(3.4) \quad \frac{d}{dx} \int_{x+1}^\infty \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda} dt.$$

The absolute value of (3.3) is bounded by

$$\int_0^1 \frac{|m'(x+t)\psi(2^{-j}(x+t))| + |2^{-j}m(x+t)\psi'(2^{-j}(x+t))|}{t^\lambda} dt.$$

This has the bound

$$C \int_0^1 \left[\frac{A}{|x+t|^\lambda t^\lambda} + \frac{2^{-j}A}{t^\lambda} \right] dt \leq CA|x|^{-\lambda} + C2^{-j}A \leq CA2^{-j\lambda}.$$

The absolute value of (3.4) is bounded by

$$|m(x+1)\psi(2^{-j}(x+1))| + C \int_{x+1}^\infty \frac{|m(t)\psi(2^{-j}t)|}{(t-x)^{\lambda+1}} dt$$

which is bounded by

$$CA2^{-j\lambda} + CA2^{-j\lambda} \int_{x+1}^\infty (t-x)^{-\lambda-1} dt \leq C2^{-j\lambda}.$$

We next estimate (3.2) for $j > 3$ and $|x| < 2^{j-2}$ or $|x| > 2^{j+2}$. If $|x| < 2^{j-2}$, the integrand in (3.2) is 0 for t not in $[2^{j-1} - x, 2^{j+1} - x]$ and a change of variables shows that the absolute value of (3.2) is

$$\left| \frac{d}{dx} \int_{2^{j-1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda} dt \right| = C \left| \int_{2^{j-1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^{\lambda+1}} dt \right|.$$

Since $|m(t)| \leq A$ and $|\psi(2^{-j}t)| \leq C$, we obtain the upper bound $CA2^{-j\lambda}$.

If $j > 3$ and $x > 2^{j+2}$, the integral in (3.2) is 0 and the estimate is trivial. If $j > 3$ and $x < 2^{j+2}$, the integrand in (3.2) is 0 for t not in $[-2^{j+1} - x, 2^{j+1} - x]$. A change of variables shows that (3.2) has absolute value bounded by

$$\left| \frac{d}{dx} \int_{-2^{j+1}}^{2^{j+1}} \frac{m(t)\psi(2^{-j}t)}{(t-x)^\lambda} dt \right| \leq C \int_{-2^{j+1}}^{2^{j+1}} \frac{|m(t)\psi(2^{-j}t)|}{|x|^{\lambda+1}} dt.$$

The facts that $|m(t)| \leq A$ and $|\psi(2^{-j}t)| \leq C$ then show that $CA2^{-j\lambda}$ is also an upper bound for this case.

To estimate the absolute value of (3.2) for $j \leq 3$, perform the differentiation to get the bound

$$\int_0^\infty \frac{|m'(x+t)\psi(2^{-j}(x+t))| + |m(x+t)2^{-j}\psi'(2^{-j}(x+t))|}{t^\lambda} dt.$$

This is less than or equal to

$$CA \int_{2^{j-1} \leq |x+t| \leq 2^{j+1}} \frac{2^{-j\lambda} + 2^{-j}}{|t|^\lambda} dt \leq C2^{-j\lambda}.$$

This completes the proof of Lemma (3.1).

LEMMA (3.5). For $0 < \lambda < 1$, assume that $|m(\lambda, x)| \leq C(1 + |x|)^{-\lambda}$ for all x and $|\partial m(\lambda, x)/\partial x| \leq C|x|^{-\lambda}$ for $|x| \leq 1$, where C is independent of x , and assume that $m(\lambda, x)$ is in $M(\infty, \lambda)$. For $\lambda > 1$ and not an integer, define $m(\lambda, x)$ recursively by

$$m(\lambda, x) = \int_{-\infty}^x g(\lambda - 1, t) dt,$$

where

$$g(\lambda - 1, t) = \frac{m(\lambda - 1, t) - \theta(t)m(\lambda - 1, 0)}{t} - \frac{\phi(t)}{t} \int_{-\infty}^{\infty} \frac{m(\lambda - 1, u) - \theta(u)m(\lambda - 1, 0)}{u} du,$$

where θ and ϕ are in C^∞ with support in $[-1, 1]$, θ is even, $\theta(0) = 1$, $\phi(0) = 0$ and $\int_{-1}^1 (\phi(t)/t) dt = 1$. Then $m(\lambda, x)$ is in $M(\infty, \lambda)$ for $\lambda > 1$ if λ is not an integer.

Note that although a version of this theorem is true for integer values of λ , a modification would be needed since $|m(1, x)| \leq C(1 + |x|)^{-1}$ for all x and $|\partial m(1, x)/\partial x| \leq C|x|^{-1}$ for $|x| \leq 1$ do not imply the integrability of $g(1, t)$ on $(-\infty, \infty)$. Since a version valid for integer values is not needed in the proof of Theorem (1.4), we will not give one here.

To prove the existence of $m(\lambda, x)$ for $\lambda > 1$, we will show inductively that

$$(3.6) \quad |m(\lambda, x)| \leq C(1 + |x|)^{-\lambda + [\lambda]}$$

and

$$(3.7) \quad \left| \frac{\partial}{\partial x} m(\lambda, x) \right| \leq C|x|^{-\lambda + [\lambda]}, \quad |x| < 1.$$

Inequalities (3.6) and (3.7) are hypothesized for $0 < \lambda < 1$. Now assume that they hold with λ replaced by $\lambda - 1$ for some $\lambda > 1$. Then for $|t| < 1$, since $-(\lambda - 1) + [\lambda - 1] = -\lambda + [\lambda]$,

$$|m(\lambda - 1, t) - m(\lambda - 1, 0)| \leq \int_0^{|t|} C|x|^{-\lambda + [\lambda]} dx \leq C|t|^{1 - \lambda + [\lambda]}.$$

This, the fact that $|\theta(t) - 1| \leq C|t|$ and (3.6) for $\lambda - 1$ imply that

$$(3.8) \quad \left| \frac{m(\lambda - 1, t) - \theta(t)m(\lambda - 1, 0)}{t} \right| \leq \begin{cases} C|t|^{-\lambda + [\lambda]}, & |t| < 1, \\ C|t|^{-\lambda - 1 + [\lambda]}, & |t| > 1. \end{cases}$$

Therefore, $g(\lambda - 1, t)$ exists and is integrable on $(-\infty, \infty)$. The estimates in (3.8) also directly imply (3.6) for $-\infty < x < 1$ and (3.7). From the definition of $g(\lambda - 1, t)$ we see that $\int_{-\infty}^{\infty} g(\lambda - 1, t) dt = 0$. Therefore $m(\lambda, x) = -\int_x^{\infty} g(\lambda - 1, t) dt$, and (3.8) implies (3.6) for $x \geq 1$. This completes the induction. Note that in particular we have $\|m(\lambda, x)\|_\infty < \infty$ for all $\lambda > 0$, λ not an integer.

We will now show inductively that $\|D^\lambda m_j(\lambda, x)\|_\infty \leq C2^{-j\lambda}$ for $\lambda > 1$ and not an integer, where C is independent of j . Therefore, fix λ , let $k = [\lambda]$ and assume that $m(\lambda - i, x)$ is in $M(\infty, \lambda - i)$ for $1 \leq i \leq k$. By the definition of $m(\lambda, x)$, $D^\lambda m_j(\lambda, x)$ is the sum of

$$(3.9) \quad 2^{-j} D^{\lambda-1} \left[m(\lambda - 1, x) \frac{\psi(2^{-j}x)}{2^{-j}x} \right],$$

$$(3.10) \quad 2^{-j} D^{\lambda-1} \left[-m(\lambda - 1, 0) \theta(x) \frac{\psi(2^{-j}x)}{2^{-j}x} \right],$$

$$(3.11) \quad -2^{-j} D^{\lambda-1} \left[\phi(x) \frac{\psi(2^{-j}x)}{2^{-j}x} \int_{-\infty}^{\infty} \frac{[m(\lambda - 1, u) - \theta(u)m(\lambda - 1, 0)]}{u} du \right]$$

and

$$(3.12) \quad 2^{-j} D^{\lambda-1} [m(\lambda, x) \psi'(2^{-j}x)].$$

Since $m(\lambda - 1, x)$ is in $M(\infty, \lambda - 1)$, Theorem (2.2) with $\phi(x)$ there taken to be $\psi(x)/x$ implies that (3.9) is bounded by $CB(m(\lambda - 1, x), \infty, \lambda - 1)2^{-j\lambda} \leq C2^{-j\lambda}$ as desired. For (3.10) and (3.11) use the fact that $\theta(x)$ and $\phi(x)$ are in $M(\infty, i)$ for all integers i . Then use Theorem (2.3) to deduce that θ and ϕ are in $M(\infty, \lambda - 1)$ and use Theorem (2.2). For (3.12) differentiate again to obtain a sum of terms like (3.9)–(3.12) with the initial 2^{-j} replaced by 2^{-2j} , $D^{\lambda-1}$ by $D^{\lambda-2}$, and ψ replaced by ψ' . These are treated in the same way; this process continues until the term

$$(3.13) \quad 2^{-jk} D^{\lambda-k} [m(\lambda, x) \psi^{(k)}(2^{-j}x)]$$

is reached. Now since $|m(\lambda - 1, x)| \leq C$, the definition of $m(\lambda, x)$ implies $|\partial m(\lambda, x)/\partial x| \leq C/|x|$ and, therefore, $m(\lambda, x)$ is in $M(\infty, 1)$. Theorem (2.3) implies that $m(\lambda, x)$ is in $M(\infty, \lambda - k)$ and Theorem (2.2) shows that (3.13) also is bounded by $C2^{-j\lambda}$. This completes the proof of Lemma (3.5).

LEMMA (3.14). *If $1 \leq s \leq \infty$, $1 < p < \infty$, k is an integer, $k \geq 0$, $k < \lambda < k + 1$, $-1 + pk < \alpha \leq -1 + p(\lambda + 1)$, $\alpha \neq -1 + p(k + 1)$, (1.3) holds for all f in $\mathcal{S}_{0,0}$ and all m in $M(s, \lambda)$ and $m(\lambda, x)$ is a set of functions as described in Lemma (3.5), then there is a constant C such that*

$$(3.15) \quad \int_{-\infty}^{\infty} |[m(\lambda - k, x) \hat{g}(x)]^\vee|^p |x|^{\alpha - kp} dx \leq C \int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha dx$$

for all g with compact support, k bounded derivatives and $\int_{-\infty}^{\infty} g(x) dx = 0$.

Since (1.3) holds for all m in $M(s, \lambda)$, it holds in particular for $m(\lambda, x)$ by Lemma (3.5) and Theorem (2.3). Now since $-1 + pk < \alpha < -1 + p(k + 2)$ and $\alpha \neq -1 + p(k + 1)$, we have $j = \text{int}((\alpha + 1)/p)$ equal to k or $k + 1$, $G(x) = x^j$ is a polynomial, $V(x) = |x|^{\alpha - jp}$ is in A_p and $W(x) = |x|^\alpha = |G(x)|^p V(x)$. Applying Theorem (2.7) to (1.3) then shows that (1.3) holds for any f in $\mathcal{Q}_k \cap L_\alpha^p$ if $m(x) = m(\lambda, x)$. In particular,

$$(3.16) \quad \int_{-\infty}^{\infty} |(m(\lambda, x)[g^{(k)}(x)]^\wedge)^\vee|^p |x|^\alpha dx \leq C \int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha dx$$

for any g of the type described by the lemma.

If $k = 0$, we are done. If $k > 0$, the left side of (3.16) is bounded below by

$$(3.17) \quad \int_{-\infty}^{\infty} \left| \left(\frac{\partial}{\partial x} m(\lambda, x) [g^{(k)}(x)]^{\wedge} \right)^{\vee} \right|^p |x|^{\alpha-p} dx$$

minus

$$(3.18) \quad C \int_{-\infty}^{\infty} \left| \left(m(\lambda, x) [xg^{(k)}(x)]^{\wedge} \right)^{\vee} \right|^p |x|^{\alpha-p} dx.$$

Now since $-1 + pk < \alpha \leq -1 + p(1 + \lambda)$ and $\alpha \neq -1 + p(k + 1)$, $(\alpha - p + 1)/p$ is not an integer and $\alpha - p \leq -1 + p\lambda < -1 + p(1 + \lambda - 1/t)$, where $t = \min(2, p', \infty)$. Furthermore, since $\int_{-\infty}^{\infty} g(x) dx = 0$, we have

$$\int_{-\infty}^{\infty} [xg^{(k)}(x)] x^j dx = 0 \quad \text{for } 0 \leq j \leq k - 1.$$

Therefore, Theorem (2.7) in conjunction with Theorem (1.2) implies that (3.18) is bounded above by the right side of (3.15) since $\text{int}((\alpha - p + 1)/p) \leq \lambda < k + 1$. This implies that (3.17) is also bounded above by the right side of (3.15).

Using the definition of $m(\lambda, x)$, we find that (3.17) is bounded below by

$$(3.19) \quad \int_{-\infty}^{\infty} \left| \left(m(\lambda - 1, x) [g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^p |x|^{\alpha-p} dx$$

minus

$$(3.20) \quad C \int_{-\infty}^{\infty} \left| \left(\theta(x) [g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^p |x|^{\alpha-p} dx$$

and

$$(3.21) \quad C \int_{-\infty}^{\infty} \left| \left(\phi(x) [g^{(k-1)}(x)]^{\wedge} \right)^{\vee} \right|^p |x|^{\alpha-p} dx.$$

Since θ and ϕ are in $M(\infty, j)$ for any integer j and $(\alpha - p + 1)/p$ is not an integer, Theorem (1.2) and Theorem (2.7) show that (3.20) and (3.21) are bounded by

$$C \int_{-\infty}^{\infty} |g^{(k-1)}(x)|^p |x|^{\alpha-p} dx,$$

and this is bounded by the right side of (3.15) by Hardy's inequality. Therefore, (3.19) is also bounded by the right side of (3.15). Repeating this procedure k times completes the proof of lemma (3.14).

To estimate one of the counterexamples, we will need the following lemma.

LEMMA (3.22). *If $-2 < b < 1$ and $\alpha = 0$ or 1, then*

$$f(x) = \left[|x|^b \sin(x^2) (\text{sgn } x)^\alpha \right]^\wedge$$

is a function and

$$(3.23) \quad |f(x)| \leq C(1 + |x|)^{\max(-1, b)}.$$

To prove this we will use the following lemma which is a simple consequence of the Leibniz alternating series theorem.

LEMMA (3.24). If $g(x) > 0$ and is monotone on $a \leq x \leq b$ and d is any real number, then

$$\left| \int_a^b \sin(x+d)g(x) dx \right| \leq \pi \int_s^{s+1} g(x) dx,$$

where $s = b - 1$ if g is increasing and $s = a$ if g is decreasing.

The following corollary of Lemma (3.24) will also be used.

LEMMA (3.25). If $g(x) > 0$ and is monotone on $a \leq x \leq b$ and d is any real number, then

$$\left| \int_a^b \sin(x+d)g(x) dx \right| \leq \pi g(s),$$

where $s = b$ if g is increasing and $s = a$ if g is decreasing.

Lemma (3.22) will be proved by showing that

$$(3.26) \quad \lim_{N \rightarrow \infty} 2 \int_0^N t^\alpha \sin(t^2) \cos\left(xt + \frac{\alpha\pi}{2}\right) dt$$

exists for every x and

$$(3.27) \quad \left| \int_0^N t^b \sin(t^2) \cos\left(xt + \frac{\alpha\pi}{2}\right) dt \right| \leq C(1 + |x|)^{\max(-1, b)},$$

where C is independent of N and x . This is sufficient since (3.26) and (3.27) imply that (3.26) equals $f(x)$ and (3.27) then shows that the absolute value of (3.26) has the bound asserted for $|f(x)|$. By symmetry, we need only prove these facts for $x \geq 0$.

To show that (3.26) exists for $x \geq 0$, we will use the identity

$$(3.28) \quad 2 \sin(t^2) \cos\left(xt + \frac{\alpha\pi}{2}\right) = \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right) + \sin\left(t^2 + xt + \frac{\alpha\pi}{2}\right).$$

If $x < M < N$, the change of variables $t - x/2 = \sqrt{u}$ shows that

$$\left| \int_M^N t^b \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right) dt \right| = \left| \int_{(M-x/2)^2}^{(N-x/2)^2} \frac{(\sqrt{u} + x/2)^b}{2\sqrt{u}} \sin\left(u - \frac{x^2}{4} - \frac{\alpha\pi}{2}\right) du \right|,$$

and since $b < 1$, Lemma (3.25) shows this is bounded by CM^{b-1} . Therefore,

$$\lim_{N \rightarrow \infty} \int_0^N t^b \sin\left(t^2 - xt - \frac{\alpha\pi}{2}\right) dt$$

exists by the Cauchy convergence criterion. The proof of the existence of the other limit obtained by substituting (3.28) in (3.26) is similar.

To prove (3.27), observe that since $b > -2$, $t^b \sin(t^2)$ has a positive derivative at 0. Let r be the smaller of $\frac{1}{4}$ and the least positive t for which $t^b \sin(t^2)$ has derivative 0. Then r depends only on b , and $t^b \sin(t^2)$ is increasing on $[0, r]$. We will now show that for $x > 1/r$ and $N > x/(1-b)$

$$(3.29) \quad \left| \int_0^r t^b \sin(t^2) \cos\left(xt + \frac{\alpha\pi}{2}\right) dt \right| \leq Cx^{-1},$$

$$(3.30) \quad \left| \int_r^{x/2} t^b \sin \left[\left(\frac{x}{2} - t \right)^2 + d \right] dt \right| \leq Cx^b + Cx^{-1},$$

$$(3.31) \quad \left| \int_{x/2}^N t^b \sin \left[\left(t - \frac{x}{2} \right)^2 + d \right] dt \right| \leq Cx^b$$

and

$$(3.32) \quad \left| \int_r^N t^b \sin \left[\left(t + \frac{x}{2} \right)^2 + d \right] dt \right| \leq Cx^{b-1} + Cx^{-1},$$

where $d = \pm \alpha\pi/2 - x^2/4$ and C is independent of N and x . This is enough to prove (3.23) for $x > 1/r$ since the left side of (3.27) is bounded by the sum of the left sides of (3.29)–(3.32).

To prove (3.29) for $x > 1/r$, make the change of variables $xt = u$ in the left side and use Lemma (3.25) to get the bound

$$\frac{\pi}{x} \left(\frac{rx}{x} \right)^b \sin \left[\left(\frac{rx}{x} \right)^2 \right],$$

and (3.29) follows.

To prove (3.30) for $x > 1/r$, make the change of variables $t = -\sqrt{u} + x/2$ in the left side to get the bound

$$\left| \frac{1}{2} \int_0^\beta g(u) \sin(u + d) du \right|,$$

where $g(u) = u^{-1/2}(-\sqrt{u} + x/2)^b$ and $\beta = (x/2 - r)^2$. Now if $g(u)$ is decreasing on $[0, \beta]$, let $s = \beta$; otherwise let s be the unique point in $[0, \beta]$ where $g(u)$ has a local minimum. By Lemma (3.24),

$$\left| \int_0^s g(u) \sin(u + d) du \right| \leq \pi \int_0^1 g(u) du$$

and this is bounded by Cx^b since $x \geq 4$. If $s < \beta$, Lemma (3.25) shows that

$$\left| \int_s^\beta g(u) \sin(u + d) du \right| \leq \pi g(\beta) = \frac{\pi r^b}{x/2 - r},$$

which is bounded by Cx^{-1} . This completes the proof of (3.30).

To prove (3.31) for $x > 1/r$, substitute $t - x/2 = \sqrt{u}$ to get

$$\left| \frac{1}{2} \int_0^{(N-x/2)^2} \left(\frac{x}{2} + \sqrt{u} \right)^b u^{-1/2} \sin(u + d) du \right|.$$

Since $b < 1$, $(x/2 + \sqrt{u})^b u^{-1/2}$ is decreasing for $u > 0$ and Lemma (3.24) completes the proof.

For (3.32) with $x > 1/r$, substitute $t + x/2 = \sqrt{u}$ to get $|\int_\beta^\gamma g(u) \sin(u + d) du|$, where $\beta = (r + x/2)^2$, $\gamma = (N + x/2)^2$ and $g(u) = \frac{1}{2} u^{-1/2} (-x/2 + \sqrt{u})^b$. If $\beta < s = [x/2(1 - b)]^2$, then $b > 0$ and g is increasing on $[\beta, s]$ and decreasing on $[s, \gamma]$. Apply Lemma (3.25) to each part; both parts have the bound $C|x|^{b-1}$. If $\beta \geq s$, g is decreasing on $[\beta, \gamma]$ and Lemma (3.25) can be applied directly to get the bound $C|x|^{-1}$. This completes the proof of Lemma (3.22) for $x > 1/r$.

To prove (3.27) for $0 \leq x < 1/r$, it is sufficient because of (3.28) to show that for $a = 2[r(1-b)]^{-1}$ and $N > a$ that

$$(3.33) \quad \left| \int_0^a t^b \sin(t^2) \cos\left(xt + \frac{\alpha\pi}{2}\right) dt \right| \leq C,$$

$$(3.34) \quad \left| \int_a^N t^b \sin\left[\left(\frac{x}{2} - t\right)^2 + d\right] dt \right| \leq C,$$

and

$$(3.35) \quad \left| \int_a^N t^b \sin\left[\left(\frac{x}{2} + t\right)^2 + d\right] dt \right| \leq C,$$

where C depends only on b .

Inequality (3.33) follows from the fact that the integrand is bounded. In (3.34) and (3.35), make the respective substitutions $t - x/2 = \sqrt{u}$, $t + x/2 = \sqrt{u}$ and apply Lemma (3.25). This completes the proof of Lemma (3.22).

4. Examples: proof of the upper bounds in Theorem (1.4). The proof that (1.3) for all f in $\mathcal{S}_{0,0}$ and m in $M(s, \lambda)$ implies the asserted upper bound for α will be done in three parts. We will first prove $\alpha \leq -1 + p(1 + \lambda - 1/s)$, next $\alpha \leq p\lambda$ and third $\alpha \leq -1 + p(\lambda + \frac{1}{2})$. The proof that $(\alpha + 1)/p$ cannot be an integer and the proof of the lower bounds are given in §5.

To prove that $\alpha \leq -1 + p(1 + \lambda - 1/s)$, let $\phi(x)$ be in C^∞ with $\phi(x) = 1$ for $|x| \leq \frac{1}{2}$, $\phi(x) = 0$ for $|x| \geq 1$ and $0 \leq \phi(x) \leq 1$ for all x . Let

$$m(x) = \sum_{k=1}^{\infty} 8^{k(1/s-\lambda)} \phi(x - 8^k),$$

and for $k \geq 2$ let

$$\hat{f}_k(x) = \phi(x8^{1-k} - 8).$$

Since $\lambda \geq 1/s$, $\|m\|_\infty \leq 1$. Because of Theorem (2.2), we may assume that the function ψ used in the definition of $B(m, s, \lambda)$ has $\psi(x) = 1$ for $\frac{3}{4} \leq |x| \leq \frac{5}{4}$. Then $m_j(x) \equiv 0$ for j not of the form $3k$ and

$$m_{3k}(x) = 2^{3k(1/s-\lambda)} \phi(x - 2^{3k}).$$

From this,

$$\|D^\lambda m_{3k}(x)\|_s = 2^{3k(1/s-\lambda)} \|(D^\lambda \phi)(x - 2^{3k})\|_s.$$

Therefore, for all $j \geq 1$,

$$\|D^\lambda m_j\|_s \leq 2^{j(1/s-\lambda)} \|D^\lambda \phi(x)\|_s.$$

Now $D^\lambda \phi$ is bounded; therefore, Lemma (2.4) implies that $\|D^\lambda \phi\|_s < \infty$. This and the definition then show that m is in $M(s, \lambda)$.

Now it is immediate that f_k is in $\mathcal{S}_{0,0}$. Since $\hat{f}_k$ has support in $[8^k - 8^{k-1}, 8^k + 8^{k-1}]$ and is 1 in $[8^k - 8^{k-2}, 8^k + 8^{k-2}]$, we have

$$m(x) \hat{f}_k(x) = 8^{k(1/s-\lambda)} \phi(x - 8^k)$$

and

$$|[m(x) \hat{f}_k(x)]^\vee| = 8^{k(1/s-\lambda)} |\check{\phi}(x)|.$$

Therefore,

$$(4.1) \quad \int_{-\infty}^{\infty} |[m(x)\hat{f}_k(x)]^\vee|^p |x|^\alpha dx = 8^{kp(1/s-\lambda)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx.$$

Similarly,

$$|f_k(x)| = 8^{k-1} |\check{\phi}(8^{k-1}x)|$$

and

$$(4.2) \quad \int_{-\infty}^{\infty} |f_k(x)|^p |x|^\alpha dx = 8^{(k-1)(p-\alpha-1)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx.$$

Using (4.1) and (4.2) in (1.3) then shows that

$$8^{kp(1/s-\lambda)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx \leq C 8^{(k-1)(p-\alpha-1)} \int_{-\infty}^{\infty} |\check{\phi}(x)|^p |x|^\alpha dx$$

for all $k \geq 0$, and this implies $\alpha \leq -1 + p(1 + \lambda - 1/s)$.

To prove that $\alpha \leq p\lambda$ for λ not an integer, fix λ , s , p and α and define

$$m(\lambda, x) = e^{-ix}/(1+x^2)^{\lambda/2}$$

for $0 < \lambda < 1$. Note that $m(\lambda, x)$ satisfies the hypothesis of Lemma (3.5), and define $m(\lambda, x)$ for $\lambda > 1$ and not an integer as was done in Lemma (3.5). We may assume that $\alpha \leq -1 + p(1 + \lambda - 1/s) \leq -1 + p(1 + \lambda)$ by the first proof of this section. We may also assume $\alpha > -1 + pk$, where $k = [\lambda]$, since otherwise $\alpha \leq -1 + pk < p\lambda$ and there is nothing to prove. As mentioned before, we also assume $(\alpha + 1)/p$ is not an integer. Now let ϕ be a C^∞ function with support in $[1, 2]$ and $\phi(x) > 0$ on $(1, 2)$ and define

$$g_n(x) = 2\phi(-2nx) - \phi(-nx).$$

Then Lemma (3.14) implies that (3.15) holds for this $m(\lambda - k, x)$ and g_n . Now

$$(4.3) \quad \int_{-\infty}^{\infty} |g_n^{(k)}(x)|^p |x|^\alpha dx = Cn^{kp-\alpha-1}.$$

Also, since $[m(\lambda - k, x)\hat{g}_n(x)]^\vee = \check{m}(\lambda - k, x) * g_n(x)$,

$$[m(\lambda - k, x)\hat{g}_n(x)]^\vee = \frac{1}{n} \int_1^2 \left[\check{m}\left(\lambda - k, x + \frac{y}{2n}\right) - \check{m}\left(\lambda - k, x + \frac{y}{n}\right) \right] \phi(y) dy.$$

By [7, p. 132], since $0 < \lambda - k < 1$,

$$\check{m}(\lambda - k, x) = A|x-1|^{\lambda-k-1} + o(|x-1|^{\lambda-k-1}),$$

where $A \neq 0$. Since $1 \leq y \leq 2$, for $1 + 1/n \leq x \leq 1 + 2/n$ we have

$$|x-1+y/2n| \leq \frac{5}{6}|x-1+y/n|.$$

It follows that

$$|x-1+y/2n|^{\lambda-k-1} - |x-1+y/n|^{\lambda-k-1} > C|x-1+y/2n|^{\lambda-k-1}$$

and, therefore,

$$|\check{m}(\lambda - k, x + y/2n) - \check{m}(\lambda - k, x + y/n)| > Cn^{k+1-\lambda} \quad \text{for large } n.$$

Thus, for n large and $1 + 1/n \leq x \leq 1 + 2/n$, we have

$$\left| [m(\lambda - k, x) \hat{g}_n(x)]^\vee \right| \geq Cn^{k-\lambda}$$

with $C > 0$. Therefore, for n large,

$$(4.4) \quad \int_{-\infty}^{\infty} \left| [m(\lambda - k, x) \hat{g}_n(x)]^\vee \right|^p |x|^{\alpha-kp} dx \geq Cn^{p(k-\lambda)-1}.$$

Using (4.3) and (4.4) in (3.15) then shows that

$$Cn^{p(k-\lambda)-1} \leq Cn^{kp-\alpha-1}$$

for sufficiently large n . We conclude that $\alpha \leq p\lambda$ for λ not an integer.

If λ is an integer, choose a sequence $\{\lambda_n\}$ with $\lambda < \lambda_n < \lambda + 1$ and $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. If (1.3) holds for all f in $\mathcal{S}_{0,0}$ and m in $M(s, \lambda)$, then by Theorem (2.3) it also holds for all f in $\mathcal{S}_{0,0}$ and m in $M(s, \lambda_n)$. By the part already proved, $\alpha \leq \lambda_n p$. Since this is true for all n , we obtain $\alpha \leq \lambda p$.

Next we shall prove that $\alpha \leq -1 + p(\lambda + \frac{1}{2})$. As in the proof that $\alpha \leq p\lambda$, we need only prove this for λ not an integer. As before, we may also assume $\alpha \leq -1 + p(1 + \lambda)$. We will also assume that $\alpha > -1 + p(\lambda + \frac{1}{2})$; from this we will derive a contradiction and conclude that $\alpha \leq -1 + p(\lambda + \frac{1}{2})$.

For $0 < \lambda < 1$, define

$$m(\lambda, x) = \left[|x|^{-2\lambda} \sin(x^2) \right]^\wedge.$$

We will use Lemma (3.14) with $k = [\lambda]$. Since $\alpha > -1 + p(\lambda + \frac{1}{2})$, we have $\alpha > -1 + pk$. We must verify that $m(\lambda, x)$ satisfies the hypotheses of Lemma (3.5). By Lemma (3.22), $|m(\lambda, x)| \leq C(1 + |x|)^{\max(-1, -2\lambda)}$. Since $\max(-1, -2\lambda) \leq -\lambda$, this implies that $|m(\lambda, x)| \leq C(1 + |x|)^{-\lambda}$. Similarly, since $\partial m(\lambda, x)/\partial x = [|x|^{1-2\lambda} \sin(x^2) \operatorname{sgn} x]^\wedge$, Lemma (3.22) implies

$$(4.5) \quad \left| \frac{\partial}{\partial x} m(\lambda, x) \right| \leq C(1 + |x|)^{1-2\lambda},$$

which is bounded for $|x| < 1$.

To show that $m(\lambda, x)$ is in $M(\infty, \lambda)$, start with the fact that $D^\lambda m(\lambda, x) = [x^\lambda |x|^{-2\lambda} \sin(x^2)]^\wedge$. By taking a linear combination of the functions in Lemma (3.22) with $b = -\lambda$, we see that

$$(4.6) \quad |D^\lambda m(\lambda, x)| \leq C(1 + |x|)^{-\lambda}$$

for all x . By Lemma (2.6) then with I equal to $[2^{j-1}, 2^{j+1}]$ or $[-2^{j+1}, -2^{j-1}]$, $\phi(x) = \chi_I(x)\psi(2^{-j}x)$ and $f(x) = m(\lambda, x)$ we get

$$\|D^\lambda \chi_I(x) m_j(\lambda, x)\|_\infty \leq C \left[\|\chi_I(x) D^\lambda m(\lambda, x)\|_\infty + 2^{-j\lambda} \|m(\lambda, x)\|_\infty \right].$$

These inequalities combined with (4.6) and the boundedness of $m(\lambda, x)$ show that $m(\lambda, x)$ is in $M(\infty, \lambda)$. Therefore, $m(\lambda, x)$ satisfies the hypothesis of Lemma (3.5).

We will now define the function $g(x)$ to be used in Lemma (3.14). Let $\phi(x)$ be a C^∞ function with support in $[0, 1]$ which is positive in $(0, 1)$. We will consider two cases. If $\alpha < -1 + p(k + 1)$, define

$$(4.7) \quad g(x) = \phi(x) - n\phi(nx).$$

The inequality $\alpha < -1 + p(k + 1)$ implies

$$(4.8) \quad \int_{-\infty}^{\infty} |g^{(k)}(x)|^p |x|^\alpha dx \leq Cn^{(k+1)p-\alpha-1}.$$

The rest of the proof will consist of showing that for n sufficiently large,

$$(4.9) \quad \int_{-\infty}^{\infty} |[m(\lambda - k, x)\hat{g}(x)]^\vee|^p |x|^{\alpha-kp} dx \geq Cn^y,$$

where $y = (k - 2\lambda)p + \alpha + 1$ and C is independent of n . Combining (3.15), (4.8) and (4.9) shows that $(k - 2\lambda)p + \alpha + 1 \leq (k + 1)p - \alpha - 1$, and this contradicts the assumption $\alpha > -1 + p(\lambda + \frac{1}{2})$.

To prove (4.9), start with the fact that for $x > 1$

$$[m(\lambda - k, x)\hat{g}(x)]^\vee = C \int_{-\infty}^{\infty} g(t) |x - t|^{2k-2\lambda} \sin(x - t)^2 dt.$$

Since $\int_{-\infty}^{\infty} g(t) dt$ is 0, the right side equals

$$(4.10) \quad C \int_{-\infty}^{\infty} g(t) [|x - t|^{2k-2\lambda} \sin(x - t)^2 - |x|^{2k-2\lambda} \sin x^2] dt.$$

Then by the definition of g , $[m(\lambda - k, x)\hat{g}(x)]^\vee$ equals a constant times the sum of

$$q(x) = \int_0^1 \phi(t) |x - t|^{2k-2\lambda} \sin(x - t)^2 dt,$$

$$r(x) = -|x|^{2k-2\lambda} \sin x^2 \int_0^1 \phi(t) dt$$

and

$$s(x) = -n \int_0^{1/n} \phi(nt) [|x - t|^{2k-2\lambda} \sin(x - t)^2 - |x|^{2k-2\lambda} \sin x^2] dt.$$

We shall show that if $0 < a < 1$ and $n > 3/a$, then

$$(4.11) \quad \int_2^{an} |q(x)|^p x^{\alpha-kp} dx \leq C_1,$$

$$(4.12) \quad \int_2^{an} |r(x)|^p x^{\alpha-kp} dx \geq C_2(an)^y$$

and

$$(4.13) \quad \int_2^{an} |s(x)|^p x^{\alpha-kp} dx \leq C_3 a^p (an)^y,$$

where C_1 , C_2 and C_3 are independent of n and a . By choosing a so that $C_3 a^p < \frac{1}{2} C_2$, it is clear that these imply (4.9) provided $y > 0$. To show that $y > 0$, compare the end terms of the inequalities

$$-1 + p(\lambda + \tfrac{1}{2}) < \alpha < -1 + p(k + 1)$$

to see that $k > \lambda - \frac{1}{2}$. This lower bound for k and the fact that $\alpha > -1 + p(\lambda + \frac{1}{2})$ imply $y > 0$.

To prove (4.11), integrate the definition of $q(x)$ by parts to get

$$q(x) = \int_0^1 \phi'(t) \int_0^t |x - u|^{2k-2\lambda} \sin(x - u)^2 du dt.$$

Assume that $x > 2$ and make the change of variables $x - u = \sqrt{w}$ in the inner integral to get

$$q(x) = \frac{1}{2} \int_0^1 \phi'(t) \int_{(x-t)^2}^{x^2} w^{k-\lambda-1/2} \sin w \, dw \, dt.$$

Now apply Lemma (3.25) to the inner integral; this shows that $|q(x)| \leq Cx^{2k-2\lambda-1}$, and (4.11) follows since $\alpha < -1 + p(k+1)$ implies $p(2k-2\lambda-1) + \alpha - kp < -1$.

Inequality (4.12) is immediate. To prove (4.13), use the facts that for $0 \leq t \leq 1/n$ and $x > 2$,

$$(4.14) \quad \left| \left[(x-t)^{2k-2\lambda} - x^{2k-2\lambda} \right] \sin(x-t)^2 \right| \leq Ctx^{2k-2\lambda-1}$$

and

$$(4.15) \quad \left| x^{2k-2\lambda} \left[\sin(x-t)^2 - \sin x^2 \right] \right| \leq Ctx^{2k-2\lambda+1}.$$

These show that $|s(x)| \leq (c/n)x^{2k-2\lambda+1}$ for $x > 2$, and (4.13) follows. This completes the proof that $\alpha \leq -1 + p(\lambda + \frac{1}{2})$ for the case $\alpha < -1 + p(k+1)$.

For the case $\alpha > -1 + p(k+1)$, let

$$g(x) = n\phi(nx) - n^2\phi(n^2x).$$

This condition on α implies that (4.8) also holds for this g and we will prove (4.9). To do this, use the fact that $[m(\lambda-k, x)\hat{g}(x)]^\vee$ equals (4.10) to write it as the sum of constants times

$$Q(x) = \int_0^{1/n} n\phi(nt) \left[|x-t|^{2k-2\lambda} - |x|^{2k-2\lambda} \right] \sin(x-t)^2 \, dt,$$

$$R(x) = |x|^{2k-2\lambda} \int_0^{1/n} n\phi(nt) \left[\sin(x-t)^2 - \sin x^2 \right] \, dt$$

and

$$S(x) = \int_0^{1/n^2} n^2\phi(n^2t) \left[|x-t|^{2k-2\lambda} \sin(x-t)^2 - |x|^{2k-2\lambda} \sin x^2 \right] \, dt.$$

The inequalities to be proved are

$$(4.16) \quad \left[\int_2^{an} |Q(x)|^p x^{\alpha-kp} \, dx \right]^{1/p} \leq \frac{C_1}{n},$$

$$(4.17) \quad \left[\int_2^{an} |R(x)|^p x^{\alpha-kp} \, dx \right]^{1/p} \geq \frac{C_2}{n} (an)^{k-2\lambda+1+(\alpha+1)/p}$$

and

$$(4.18) \quad \left[\int_2^{an} |S(x)|^p x^{\alpha-kp} \, dx \right]^{1/p} \leq \frac{C_3}{n^2} (an)^{k-2\lambda+1+(\alpha+1)/p}$$

for $0 < a < 1$ and $n > 3/a$; as in the last case these are sufficient to prove (4.9) since $\alpha > -1 + p(k+1)$ implies $k-2\lambda+1+(\alpha+1)/p > 0$.

For (4.16), use (4.14) to show that $|Q(x)| \leq (C/n)x^{2k-2\lambda-1}$ for $x > 2$. This and the fact that $\alpha < -1 + p(\lambda+1)$ imply (4.16). For (4.17), use a trigonometric identity to show that

$$R(x) = -|x|^{2k-2\lambda} \int_0^{1/n} n\phi(nt) 2 \cos\left(x^2 - xt + \frac{t^2}{2}\right) \sin\left(xt - \frac{t^2}{2}\right) \, dt.$$

Now if $0 \leq t \leq 1/n$ and $1 \leq x \leq n$, then $\sin(xt - t^2/2) \geq Cxt$, and if we also have $2j\pi \leq x^2 \leq (2j + \frac{1}{3})\pi$, then $\cos(x^2 - xt + t^2/2) \geq C > 0$. Therefore, if $j \geq 1$ and $\sqrt{2j\pi} \leq x \leq \sqrt{(2j + \frac{1}{3})\pi} \leq n$, we have $|R(x)| \geq (C/n)x^{2k-2\lambda+1}$. Inequality (4.17) follows from this. For (4.18), use (4.14) and (4.15) to show that $|S(x)| \leq (C/n^2)x^{2k-2\lambda+1}$ for $|x| \geq 2$. This completes the proof that $\alpha \leq -1 + \lambda(p + \frac{1}{2})$ and completes the proof of the upper bounds in Theorem (1.4).

5. Completion of the proof of Theorem (1.4). This will be done in three parts. The proof that $\alpha > -1$ is done directly. This result and a duality argument based on the upper bounds proved in §4 then prove that $\alpha \geq \max(-p\lambda, -1 + p(-\lambda + \frac{1}{2}))$. Finally, we show that $(\alpha + 1)/p$ cannot be a positive integer.

To prove $\alpha > -1$, we will show for every integer k that $-kp < \alpha \leq -1$ is impossible. To do this, fix k and α and choose g in C^∞ with support in $[1, 2]$ such that $\int_1^2 g(x) dx = 1$ and $\int_1^2 x^j g(x) dx = 0$ for $1 \leq j \leq k$; the existence of such a g is shown in Lemma 2.6, p. 182, of [1]. Define $f(x) = \check{g}(x) - \check{g}(-x)$. Then $f(x)$ is in $\mathcal{S}_{0,0}$ and $f^{(j)}(0) = 0$ for $0 \leq j \leq k$. Therefore,

$$(5.1) \quad \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx < \infty.$$

Since (1.3) is assumed to hold for f in $\mathcal{S}_{0,0}$ and all m in $M(s, \lambda)$, it holds in particular for this $f(x)$ and $m(x) = i \operatorname{sgn} x$. Therefore, (1.3) and (5.1) imply that

$$(5.2) \quad \int_{-\infty}^{\infty} |\tilde{f}(x)|^p |x|^\alpha dx < \infty,$$

where

$$\tilde{f}(x) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0+} \int_{|t|>\epsilon} \frac{f(x-t)}{t} dt$$

is the Hilbert transform of f .

We will now show that (5.2) cannot hold. To do this, define

$$G(x) = \int_{-\infty}^x [g(t) - g(-t)] dt.$$

Since g has support in $[1, 2]$ and $g(t) - g(-t)$ has integral 0, $G(x)$ has support in $[-2, 2]$. Therefore, G is integrable and

$$x\check{G}(x) = i[\check{g}(x) - \check{g}(-x)] = if(x).$$

From this we see that

$$G(0) = i \int_{-\infty}^{\infty} \frac{f(t)}{t} dt$$

and

$$\tilde{f}(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(-t)}{t} dt = \frac{i}{\pi} G(0) = \frac{-i}{\pi}.$$

Since $\tilde{f}$ is in C^∞ and $\alpha \leq -1$, this shows that $\int_{-\infty}^{\infty} |\tilde{f}(x)|^p |x|^\alpha dx = \infty$ and contradicts (5.2). This completes the proof that $\alpha > -1$.

To prove that $\alpha \geq \max(-p\lambda, -1 + p(-\lambda + \frac{1}{2}))$, we may assume by the previous part that $\alpha > -1$. We may also assume $\alpha < p - 1$ since if $\alpha \geq p - 1$ there is nothing to prove. Now since (1.3) holds for all f in $\mathcal{S}_{0,0}$, Theorem (2.7) shows that (1.3) holds for all f in $L^2 \cap L_\alpha^p$. Now if g is any function in $\mathcal{S}_{0,0}$,

$$(5.3) \quad \left[\int_{-\infty}^{\infty} |[m(x)\hat{g}(x)]^\vee|^{p'} |x|^{-\alpha/(p-1)} dx \right]^{1/p'} \\ = \sup_{f \in A} \left| \int_{-\infty}^{\infty} [m(x)\hat{g}(x)]^\vee f(x) dx \right|,$$

where A is the set of all f in L^2 with $\int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx \leq 1$. Since f and g are in L^2 , the right side equals

$$\sup_{f \in A} \left| \int_{-\infty}^{\infty} g(x) [m(-x)\hat{f}(x)]^\vee dx \right|.$$

Now use Hölder's inequality and (1.3) to conclude that

$$\int_{-\infty}^{\infty} |[m(x)\hat{g}(x)]^\vee|^{p'} |x|^{-\alpha/(p-1)} dx \leq C \int_{-\infty}^{\infty} |g(x)|^{p'} |x|^{-\alpha/(p-1)} dx.$$

Since this is true for all g in $\mathcal{S}_{0,0}$ and m in $M(s, \lambda)$, the proof in §4 shows that

$$-\alpha/(p-1) \leq \min[p'\lambda, -1 + p'(\lambda + \frac{1}{2})].$$

Multiplying by $1 - p$ then gives the asserted inequality.

Finally, we show that $(\alpha + 1)/p$ can not be a positive integer. Since $m(x) = \text{sgn } x$ is in $M(s, \lambda)$ for all $\lambda > 0$ and s satisfying $1 \leq s \leq \infty$, it is sufficient to show that (1.3) cannot hold for this m if $(\alpha + 1)/p$ is a positive integer. To do this, fix p satisfying $1 < p < \infty$ and a positive integer k . Let $\alpha = -1 + pk$ and $m(x) = \text{sgn } x$ and assume that (1.3) holds for all f in $\mathcal{S}_{0,0}$. We then have for all f in $\mathcal{S}_{0,0}$

$$(5.4) \quad \int_{-\infty}^{\infty} |\tilde{f}(x)|^p |x|^\alpha dx \leq C \int_{-\infty}^{\infty} |f(x)|^p |x|^\alpha dx.$$

By Theorem (2.7), (5.4) will also hold for all f in $L_\alpha^p \cap Q_{k-2}$.

Now choose $f(x)$ in $L^\infty \cap Q_{k-2}$ with support in $[0, 1]$ such that

$$(5.5) \quad \int_{-\infty}^{\infty} x^{k-1} f(x) dx = 1;$$

such an f exists by Lemma 2.6, p. 182, of [1]. Now since f is in Q_{k-2} , we see that

$$\int_{-\infty}^{\infty} \frac{x^{k-1} - t^{k-1}}{x - t} f(t) dt = 0;$$

therefore, for $x > 2$

$$x^{k-1} \tilde{f}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t^{k-1} f(t)}{x - t} dt.$$

This and (5.5) show that for x large we have $|\tilde{f}(x)| \geq x^{-k}/2\pi$ and that, therefore, the left side of (5.4) is infinite. Since the right side of (5.4) is finite, we have a contradiction. This completes the proof that $(\alpha + 1)/p$ is not a positive integer.

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